Classical Free Fall Calculation

This Mathematica Notebook performs the calculation of the time of free fall from the orbit of the Earth to the Sun, using simple Newtonian functions.

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In[1]:=
```
Clear[GM, Φ, r, t, m, v, A]
```

The Newtonian Acceleration of a particle in free fall is the derivative of the potential

In[2]:=
```
Φ[r_] := GM / r
```

During Free Fall, energy is conserved, so falling at rest from A to r yields

In[3]:=
```
SetAttributes[A, Constant]
```

In[4]:=
```
Energy = m v^2 / 2 - m Φ[r] == - m Φ[A]
```

Out[4]= $-\frac{GM\,m}{r} + \frac{m\,v^2}{2} == -\frac{GM\,m}{A}$

In[5]:=
```
V = Solve[ Energy, v]
```

Out[5]= $\left\{\left\{v \to -\sqrt{2}\sqrt{-\frac{GM}{A} + \frac{GM}{r}}\right\}, \left\{v \to \sqrt{2}\sqrt{-\frac{GM}{A} + \frac{GM}{r}}\right\}\right\}$

In[6]:=
```
f = 1 / (v /. V[[1]])
```

Out[6]= $-\frac{1}{\sqrt{2}\sqrt{-\frac{GM}{A} + \frac{GM}{r}}}$

Now work with the special case of falling from rest at 1 AU towards the Sun

In[7]:=
```
A = 1
```

Out[7]= 1

Calculate the function, T[r], the time it takes a particle to fall from rest at 1 AU, to a ratius r. The Integral function of a freely falling particle is surprisingly complicated:

```
In[8]:= T = Integrate[f, r]
```

Out[8]= $-\frac{(-1+r)\sqrt{r}+\sqrt{-1+r}\,\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]}{\sqrt{2}\sqrt{GM\left(-1+\frac{1}{r}\right)}\sqrt{r}}$

Now calculate the duration of the fall from the limits at one AU and zero AU.

```
In[9]:= AB = Simplify[Limit[T, r → 1]]
```

Out[9]= 0

```
In[10]:= AA = Simplify[Limit[T, r → 0]]
```

Out[10]= $\frac{\pi}{2\sqrt{2}\sqrt{GM}}$

```
In[11]:= N[AA]
```

Out[11]= $\frac{1.11072}{\sqrt{GM}}$

Calculation Units: Conversion to Days

The Distance Units for this calculation are AU. The time units for this calculation are days. In these units, GM is a small number, indicating that the system is not relativistic. GC = gravitational constant = 6.67300 × 10^−11 m3 kg^−1 s^−1; Astronomical Unit = 149 598 000 000; One Day = 86400 s

```
In[12]:= GC = 6.673 10^-11 m^3 / kg / s^2
```

Out[12]= $\frac{6.673\times10^{-11}\ m^3}{kg\ s^2}$

```
In[13]:= MS = 1.9891 10^30 kg
```

Out[13]= 1.9891×10^{30} kg

```
In[14]:= AU = 1.49598 10^11 m
```

Out[14]= 1.49598×10^{11} m

```
In[15]:= day = 3600 24 s
```

Out[15]= 86400 s

Calculate the Speed of Light in AU per day (Its a fairly small number)

```
In[40]:= clight = (299782458 m / s) (day / AU)
```

Out[40]= 173.139

The Distance Units for this calculation are AU. The time units for this calculation are days. In these units, GM (gm) is a small number, indicating that our solar system is not general−relativistic.

```
In[17]:= gm = GC MS day^2 / AU^3

Out[17]= 0.000295956
```

Formally, the velocity of the particle goes to infinity as r goes to zero. To avoid this, the calculation will stop at the Solar Radius (RS). The Sun can not be treated as a point source inside a Solar Radius anyway.

```
In[18]:= RS = 6.95987 10^8 m
```

Out[18]= 6.95987×10^{8} m

In Astronomical Units (AU) the Solar Radius is 0.47 % of an AU.

```
In[19]:= rs = RS / AU

Out[19]= 0.00465238

In[20]:= MaxT = Limit[ AA, GM -> gm]

        General::spell1 :
         Possible spelling error: new symbol name "MaxT" is similar to existing symbol "Max". More…

Out[20]= 64.5641
```

Now numerically evaluate the time, T in days,
it takes an object to fall from rest from 1 AU to the surface of the Sun.

```
In[21]:= MaxT = Abs[Limit[ Limit[T, r → rs], GM -> gm]]

Out[21]= 64.5554
```

Calculation of the Speed: v[r]

```
In[22]:= T
```

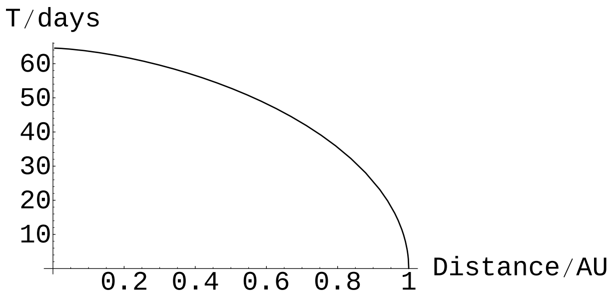
Out[22]= $$-\frac{(-1+r)\sqrt{r}+\sqrt{-1+r}\,\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]}{\sqrt{2}\sqrt{\mathrm{GM}\left(-1+\frac{1}{r}\right)}\sqrt{r}}$$

t[r] is the time to fall a distance r(AU) in units of days

```
In[23]:= t = Limit[ T, GM → gm]
```

Out[23]= $$\frac{-(-1+r)\sqrt{r}-\sqrt{-1+r}\,\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]}{\sqrt{-0.000591913+\frac{0.000591913}{r}}\sqrt{r}}$$

```
In[26]:= Plot[t, {r, rs, 1}, AxesLabel → {"Distance/AU", "T/days"}]
```




```
Out[26]= - Graphics -
```

Calculate the speed of the falling object in units of the speed of light.

```
In[28]:= OneOverV = D[t, r]
```

Out[28]=

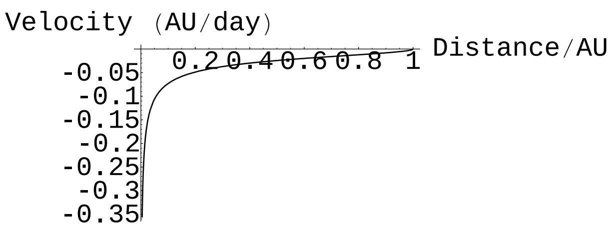
$$\frac{-\frac{\left(\frac{1}{2\sqrt{-1+r}}+\frac{1}{2\sqrt{r}}\right)\sqrt{-1+r}}{\sqrt{-1+r}+\sqrt{r}}-\frac{-1+r}{2\sqrt{r}}-\sqrt{r}-\frac{\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]}{2\sqrt{-1+r}}}{\sqrt{-0.000591913+\frac{0.000591913}{r}}\;\sqrt{r}}+\frac{0.000295956\left(-(-1+r)\sqrt{r}-\sqrt{-1+r}\;\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]\right)}{\left(-0.000591913+\frac{0.000591913}{r}\right)^{3/2}r^{5/2}}-\frac{-(-1+r)\sqrt{r}-\sqrt{-1+r}\;\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]}{2\sqrt{-0.000591913+\frac{0.000591913}{r}}\;r^{3/2}}$$

```
In[29]:= v = 1 / OneOverV
```

Out[29]=

$$1\Bigg/\left(\frac{-\frac{\left(\frac{1}{2\sqrt{-1+r}}+\frac{1}{2\sqrt{r}}\right)\sqrt{-1+r}}{\sqrt{-1+r}+\sqrt{r}}-\frac{-1+r}{2\sqrt{r}}-\sqrt{r}-\frac{\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]}{2\sqrt{-1+r}}}{\sqrt{-0.000591913+\frac{0.000591913}{r}}\;\sqrt{r}}+\frac{0.000295956\left(-(-1+r)\sqrt{r}-\sqrt{-1+r}\;\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]\right)}{\left(-0.000591913+\frac{0.000591913}{r}\right)^{3/2}r^{5/2}}-\frac{-(-1+r)\sqrt{r}-\sqrt{-1+r}\;\mathrm{Log}\left[\sqrt{-1+r}+\sqrt{r}\right]}{2\sqrt{-0.000591913+\frac{0.000591913}{r}}\;r^{3/2}}\right)$$

```
In[35]:= Plot[v, {r, rs, 1}, AxesLabel → {"Distance/AU", "Velocity (AU/day)"}]
```

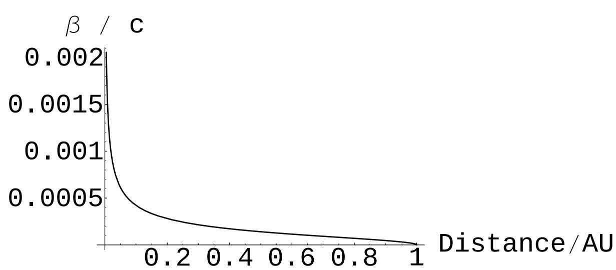



```
Out[35]= - Graphics -
```

The velocity is negative because it is the infall velocity. The magnitude of the velocity is :

```
In[41]:= mag := Abs[v] / clight

In[42]:= Plot[mag, {r, rs, 1}, AxesLabel → {"Distance/AU", "β / c"}]
```




```
Out[42]= - Graphics -
```

The maximum velocity is the velocity when reaching the surface of the Sun (in units of c).

```
In[46]:= MaxMag = Abs[Limit[mag, r → rs]]

Out[46]= 0.00205534
```

Converting back to km/s

```
In[48]:= MaxMag 299782.458 km / s
```

Out[48]= $\frac{616.156\ \text{km}}{\text{s}}$